PHYS 102 – General Physics II Final Exam Solution

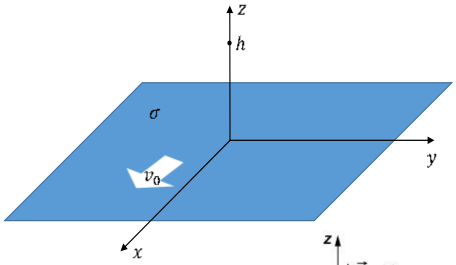
Duration: 150 minutes — Monday, 5 June 2023; 09:30

1. An infinite plane of non-conducting material is charged with uniform surface charge density $+\sigma$. Take the plane to be the xy-plane (i.e., $z = 0$). If it is moving with constant velocity $\vec{\mathbf{v}}_0 = v_0\hat{\mathbf{i}}$ in the x direction:

(a) (6 Pts.) find the electric field vector at the point $(x = 0, y = 0, z = h)$;

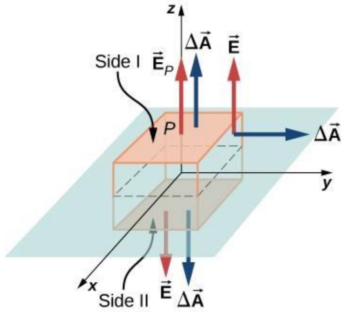
(b) (7 Pts.) find the magnetic field vector at the same point;

(c) (7 Pts.) find the Poynting vector $\vec{\mathbf{S}}$ at the same point.



Solution:

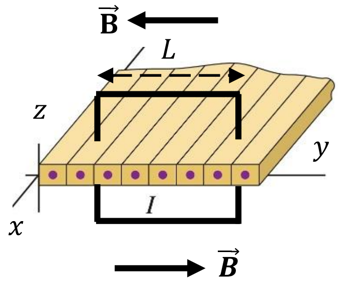
(a) Applying Gaus's law to the closed surface shown, we get



$$\oiint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{Q_{\text{enc}}}{\epsilon_0} \quad \rightarrow \quad 2EA = \frac{\sigma A}{\epsilon_0} \quad \rightarrow \quad \vec{\mathbf{E}} = \frac{\sigma}{2\epsilon_0}\hat{\mathbf{k}}\,.$$

(b) Solution can be obtained using either Ampère's law or Biot-Savart law.

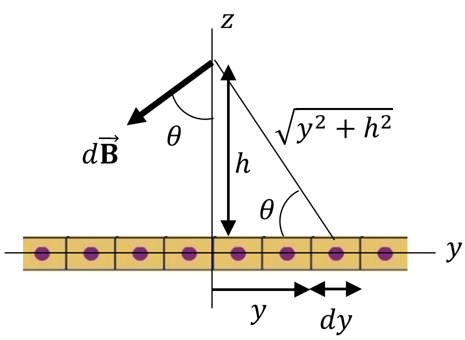
(b1) Applying Ampère's law to the rectangular closed loop shown in the figure, we have



$$dq = \sigma\, dx\, dy\,, \qquad \frac{dq}{dt} = \sigma\frac{dx}{dt}\,dy \quad \rightarrow \quad dI = \sigma v_0\, dy$$

$$\oint \vec{\mathbf{B}} \cdot d\vec{\boldsymbol{\ell}} = \mu_0 I_{\text{enc}} \quad \rightarrow \quad 2B_y L = -\mu_0\sigma v_0 L \quad \rightarrow \quad B_y = -\frac{1}{2}\mu_0\sigma v_0\,.$$

(b2) Dividing the sheet into infinitesimal lines of current, we can use Biot-Savart law. From translational symmetry of the current in the y direction, we know that the magnetic field is only in the $\pm y$ direction.



$$dB = \frac{\mu_0\, dI}{2\pi\sqrt{y^2 + h^2}}\,, \qquad dB_y = -dB\sin\theta$$

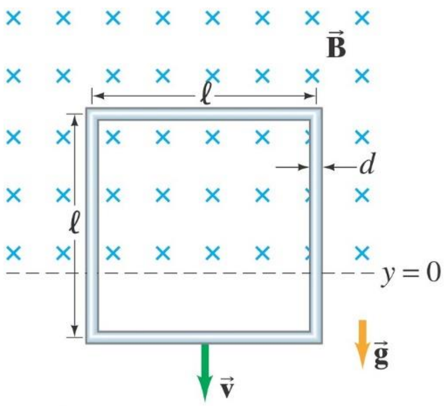
$$\sin\theta = \frac{h}{\sqrt{y^2 + h^2}} \quad \rightarrow \quad dB_y = -\frac{\mu_0\sigma v_0 h}{2\pi}\frac{dy}{y^2 + h^2}$$

$$B_y = -\frac{\mu_0\sigma v_0 h}{2\pi}\int_{-\infty}^{\infty}\frac{dy}{y^2 + h^2} = -\frac{\mu_0\sigma v_0 h}{2\pi}\left(\frac{\pi}{h}\right) \quad \rightarrow \quad \vec{\mathbf{B}} = -\frac{1}{2}\mu_0\sigma v_0\,\hat{\mathbf{j}}\,.$$

(c)

$$\vec{\mathbf{S}} = \frac{1}{\mu_0}\vec{\mathbf{E}} \times \vec{\mathbf{B}} \quad \rightarrow \quad \vec{\mathbf{S}} = \left(\frac{\sigma^2 v_0}{4\epsilon_0}\right)\hat{\mathbf{i}}\,.$$

2. In a certain region of space near Earth's surface, a uniform horizontal magnetic field of magnitude B exists above a level defined to be $y = 0$. Below $y = 0$, the field abruptly becomes zero. A vertical square wire loop has mass m, resistivity ρ, diameter d, and side length ℓ. It is initially at rest with its lower horizontal side at $y = 0$ and is then allowed to fall under gravity, with its plane perpendicular to the direction of the magnetic field, which is into the plane of the figure. Assume that $d \ll \ell$, so $\ell \pm d \approx \ell$.



(a) (4 Pts.) What is the direction (clockwise or counter clockwise) of the current induced in the loop? Why?

(b) (8 Pts.) While the loop is still partially immersed in the magnetic field (as it falls into the zero-field region), determine the magnetic "drag" force that acts on it at the moment when its speed is v. What is the direction of the force?

(c) (8 Pts.) Assume that the loop achieves a terminal velocity v_T before its upper horizontal side exits the field. Determine a formula for v_T.

Solution: (a) As the loop falls out of the magnetic field, the flux through the loop decreases with time creating an induced emf in the loop. By Lenz's law, the induced current should be **clockwise** so that its magnetic field should oppose the decrease of the flux.

(b) Assume that at time t when the speed of the loop is $v(t)$, the upper horizontal side is at y. Flux of the magnetic field through the loop is

$$\Phi_B = B\ell y \quad \rightarrow \quad \frac{d\Phi_B}{dt} = B\ell\frac{dy}{dt} = B\ell v(t) \quad \rightarrow \quad |\mathcal{E}| = \frac{d\Phi_B}{dt} = B\ell v(t)\,.$$

The current in the loop is equal to the induced emf divided by the resistance, which can be written in terms of the resistivity as

$$R = \frac{\rho(4\ell)}{\left(\frac{\pi d}{2}\right)^2} = \frac{16\rho\ell}{\pi d^2} \quad \rightarrow \quad I = \frac{|\mathcal{E}|}{R} = \frac{\pi d^2 B}{16\rho}v(t)\,.$$

This current induces a force on the three sides of the loop in the magnetic field. The forces on the two vertical sides are equal and opposite and therefore cancel. The force on the horizontal side is

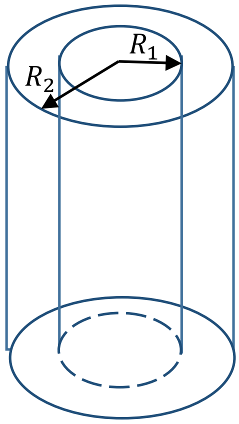
$$F_B = I\ell B \quad \rightarrow \quad F_B = \frac{\pi d^2 B^2 \ell}{16\rho}v(t)\,.$$

By Lenz's law this force is upward to slow the decrease in flux.

(c) Terminal speed v_T will occur when the gravitational force is equal to the magnetic force.

$$F_B = mg \quad \rightarrow \quad \frac{\pi d^2 B^2 \ell}{16\rho}v_T = mg \quad \rightarrow v_T = \frac{16\rho m g}{\pi d^2 B^2 \ell}\,.$$

3. Consider a coaxial cable made from two thin hollow coaxial tubes. The inner conductor has radius R_1 and the outer conductor has radius R_2.



(a) (10 Pts.) Find the capacitance per unit length.

(b) (10 Pts.) Find the inductance per unit length.

Solution:

(a) To find the capacitance per unit length, we first use Gauss's law to find the electric field between the two conductors using a concentric cylindrical Gaussian surface of radius r situated between the two conductors, i.e., $R_1 < r < R_2$.

$$\oiint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{Q_{\text{enc}}}{\epsilon_0} \quad \rightarrow \quad E(r) = \frac{Q}{2\pi\epsilon_0 \ell r}$$

$$|\Delta V| = \int_{R_1}^{R_2} E(r)dr = \frac{Q}{2\pi\epsilon_0 \ell} \int_{R_1}^{R_2} \frac{dr}{r} \quad \rightarrow \quad \Delta V = \frac{Q}{2\pi\epsilon_0 \ell} \ln\left(\frac{R_2}{R_1}\right)$$

$$Q = C\,\Delta V \quad \rightarrow \quad C = \frac{Q}{\Delta V} \quad \rightarrow \quad C = \frac{2\pi\epsilon_0 \ell}{\ln(R_2/R_1)} \quad \rightarrow \quad \frac{C}{\ell} = \frac{2\pi\epsilon_0}{\ln(R_2/R_1)}.$$

(b) To find inductance per unit length, we first use Ampère's law to find the magnetic field between the two conductors where the path is circle of radius r, ($R_1 < r < R_2$).

$$\oint \vec{\mathbf{B}} \cdot d\vec{\boldsymbol{\ell}} = B(r)(2\pi r) = \mu_0 I \quad \rightarrow \quad B(r) = \frac{\mu_0 I}{2\pi r}, \qquad R_1 < r < R_2\,.$$

The magnetic flux through a rectangle of width dr and length ℓ along the cable, a distance r from the center is

$$d\Phi_B = B(r)\,(\ell dr) = \frac{\mu_0 I \ell dr}{2\pi r} \quad \rightarrow \quad \Phi_B = \int d\Phi_B = \frac{\mu_0 I \ell}{2\pi} \int_{R_1}^{R_2} \frac{dr}{r} \quad \rightarrow \quad \Phi_B = \frac{\mu_0 I \ell}{2\pi} \ln\left(\frac{R_2}{R_1}\right).$$

Therefore,

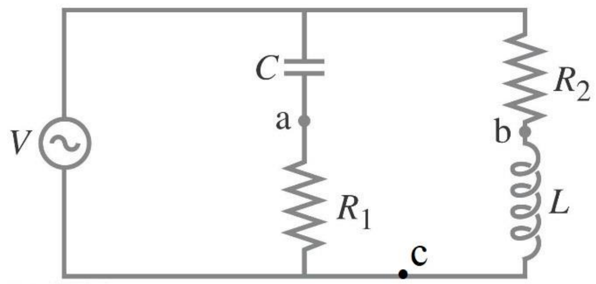
$$L = \frac{\Phi_B}{I} = \frac{\mu_0 \ell}{2\pi} \ln\left(\frac{R_2}{R_1}\right) \quad \rightarrow \quad \frac{L}{\ell} = \frac{\mu_0}{2\pi} \ln\left(\frac{R_2}{R_1}\right).$$

4. Consider the circuit shown where the voltage of the AC source is given as $V = V_0 \cos(\omega t)$.

(a) (6 Pts.) Find the rms voltage V_{ac} across the resistance R_1.

(b) (6 Pts.) Find the rms voltage V_{bc} across the inductor L.

(c) (8 Pts.) Find R_1 in terms of $R_2, L,$ and C so that the potential difference $V_{\text{ac}} - V_{\text{bc}}$ is zero for all frequencies.



Solution:

With the given applied voltage, the rms currents through the capacior and the inductor branches are

$$I_{C\,rms} = \frac{V_{rms}}{\sqrt{R_1^2 + X_C^2}}\,, \qquad I_{L\,rms} = \frac{V_{rms}}{\sqrt{R_2^2 + X_L^2}}\,.$$

(a) Therefore, the rms voltage across the resistor R_1 is

$$V_{\text{ac}} = I_{C\,rms} R_1 = \frac{V_0 R_1}{\sqrt{2}\sqrt{R_1^2 + X_C^2}}.$$

(b) Similarly, the rms voltage across the inductor is

$$V_{\text{bc}} = I_{L\,rms} X_L = \frac{V_0 X_L}{\sqrt{2}\sqrt{R_2^2 + X_L^2}}\,.$$

(c)

$$V_{\text{ac}} - V_{\text{bc}} = 0 \quad \rightarrow \quad \frac{V_0 R_1}{\sqrt{2}\sqrt{R_1^2 + X_C^2}} = \frac{V_0 X_L}{\sqrt{2}\sqrt{R_2^2 + X_L^2}} \quad \rightarrow \quad \frac{R_1}{\sqrt{R_1^2 + X_C^2}} = \frac{X_L}{\sqrt{R_2^2 + X_L^2}}\,.$$

Squaring both sides, we get

$$\frac{R_1^2}{R_1^2 + X_C^2} = \frac{X_L^2}{R_2^2 + X_L^2} \quad \rightarrow \quad R_1^2 R_2^2 + R_1^2 X_L^2 = R_1^2 X_L^2 + X_L^2 X_C^2 \quad \rightarrow \quad R_1^2 R_2^2 = X_L^2 X_C^2 \quad \rightarrow \quad R_1 R_2 = X_L X_C\,.$$

Finally, using $X_L = \omega L$ and $X_C = 1/\omega C$, we find

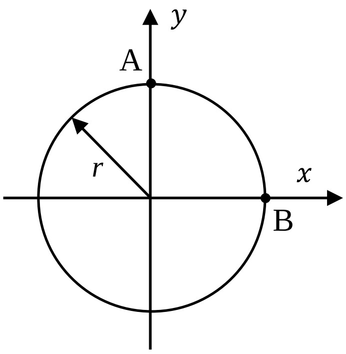
$$R_1 R_2 = \frac{L}{C} \quad \rightarrow \quad R_1 = \frac{L}{R_2 C}\,,$$

which is independent of the frequency ω.

5. In a circular region of space on the xy-plane there exists a uniform magnetic field which changes in time according to the expression $\vec{\mathbf{B}} = B_0(1 - e^{-t/\tau})\,\hat{\mathbf{k}}$, where B_0 and τ are positive constants.

(a) (10 Pts.) Find the expression for the magnitude of the electric field at point A of the figure. Draw the direction of the electric field vector at point A on the figure.

(b) (10 Pts.) Draw the direction of the Poynting vector at point B of the figure. What is its magnitude?



Solution:

(*a*) Magnetic flux through the circular region enclosed by the circle C of radius r is

$$\Phi_B = B(\pi r^2) \;\rightarrow\; \Phi_B = \pi r^2 B_0(1 - e^{-t/\tau}).$$

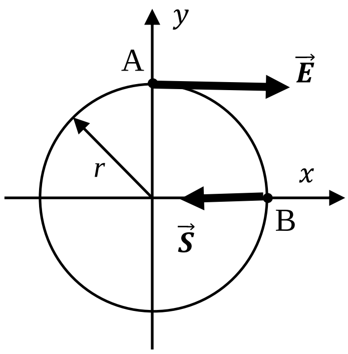
The electromotive force induced arount the circle is

$$|\mathcal{E}| = \frac{d\Phi_B}{dt} \;\rightarrow\; \frac{1}{\tau}\pi r^2 B_0 e^{-t/\tau}.$$

Since

$$|\mathcal{E}| = \oint \vec{\mathbf{E}} \cdot d\vec{\boldsymbol{\ell}} = |E|(2\pi r) \;\rightarrow\; |E|(2\pi r) = \frac{1}{\tau}\pi r^2 B_0 e^{-t/\tau} \;\rightarrow\; |E| = \frac{B_0}{2\tau} r e^{-t/\tau}.$$

Magnetic flux is increasing, therefore, according to Lenz's law, the electric field induced on the circle C is in clockwise direction.



(b)

$$\vec{\mathbf{S}} = \frac{1}{\mu_0}\vec{\mathbf{E}} \times \vec{\mathbf{B}}$$

The electric and magnetic fields are perpendicular to each other. Therefore,

$$S = \frac{B_0^2 r}{2\mu_0 \tau} e^{-t/\tau}(1 - e^{-t/\tau}),$$

its direction being $-\hat{\mathbf{j}} \times \hat{\mathbf{k}} = -\hat{\mathbf{i}}$, i.e., inward at the point B.